Repetitive Flutter Calculations in Structural Design

Raphael T. Haftka*
Illinois Institute of Technology, Chicago, Ill.
and
E. Carson Yates Jr.†
NASA Langley Research Center, Hampton, Va.

Some aspects of efficient modal flutter analysis are investigated for use in aircraft structural design which may involve many iterations. Expressions for the generalized aerodynamic forces are derived which are separated into mode-dependent and mode-independent parts; this permits rapid recalculation of the forces when the modes are changed. The computer times required for the various parts of a single flutter analysis are presented for some example problems. These examples are used to compare the efficiency of using periodically updated natural vibration modes fixed modes with resizing methods that do not require the derivatives of any of the flutter parameters with respect to structural variables. The main computational penalty in updating modes is found to be in the recalculation of the modes, rather than in the calculation of the generalized aerodynamic forces. Flutter calculations also are examined for resizing methods that do require the derivatives of the flutter frequency, flutter speed, or flutter altitude with respect to design variables. The convergence of such derivatives with increasing number of modes is investigated with the aid of two examples. The poor convergence of the derivatives precluded comparison of the use of continually updated vs fixed modes for resizing methods that require such derivatives.

Nomenclature

$\bar{A}$	= aerodynamic influence coefficient (AIC) matrix (structural grid)
A	= generalized-aerodynamic-force (GAF) matrix
a	= speed of sound
B	= matrix of pressure-kernel integrals; components defined by Eq. (A12)
B_l	= defined by Eq. (A6)
b	= root semichord
c	= vector of flutter modal amplitudes [Eq. (2), right flutter eigenvector]
c^*	= left flutter eigenvector
E	= matrix defined by Eq. (A11)
e_l	= coefficient of pressure function, see. Eq. (A4)
$\bar{G}$	= structural damping matrix (structural degrees of freedom)
G	= modal structural damping matrix
H	= flutter altitude
H	= matrix defined by Eq. (A17)
$h^i(x,y)$	= z displacement at point (x,y) per unit amplitude of ith mode
I	= unit matrix
$\bar{K}$	= stiffness matrix (structural degrees of freedom)
K	= modal stiffness matrix
k	= reduced frequency, $b\omega/V$
L_k	= defined by Eq. (9)
L_M	= defined by Eq. (10)
L_ρ	= defined by Eq. (11)
$\bar{M}$	= mass matrix (structural degrees of freedom)
M	= modal mass matrix
M	= Mach number
m	= number of structural degrees of freedom
n	= number of vibration modes
n_c	= number of collocation points
n_p	= number of pressure functions
n_q	= number of quadrature points
P	= matrix defined by Eq. (A9)
q	= dynamic pressure, $\frac{1}{2}\rho V^2$
q_d	= design dynamic pressure for flutter
R	= defined by Eq. (12)
U	= matrix defined by Eq. (A16)
u_l	= lth pressure function
V	= freestream speed
W	= matrix defined by Eq. (A10)
$\bar{w}$	= downwash
x	= chordwise coordinate (dimensional)
y	= spanwise coordinate (dimensional)
Z	= matrix of structural modes
Δp	= lifting pressure
ρ	= freestream air mass density
ω	= frequency
ω_i	= frequency of ith vibration mode
Subscript	
F	= pertaining to flutter

Based in part on Paper 74-141 presented at the AIAA 12th Aerospace Sciences Meeting, Washington, D.C., Jan. 30-Feb. 1, 1974; received June 16, 1975; revision received Sept. 29, 1975. A portion of this research was supported by NASA grant NGR 52-012-008.

Index categories: Structural Design, Optimal; Aeroelasticity and Hydroelasticity.

*Assistant Professor, Department of Mechanics, Mechanical and Aerospace Engineering. Member AIAA.

†Aerospace Engineer, Structures & Dynamics Divison. Associate Fellow AIAA.

Introduction

IN recent years, considerable effort has been directed toward development of computer-aided procedures for the design of minimum-mass aircraft structures that satisfy multiple, interdisciplinary requirements such as strength, flutter, divergence, and aeroelastic deformations. These procedures require that unsteady aerodynamic forces and flutter solutions be re-evaluated and structural members be resized many times. For such processes to be computationally economical, the repeated portions of the calculations must be minimized and performed in the most efficient manner possible. This kind of capability is needed, especially in automated design systems, so that aeroelastic phenomena may be studied in depth and potential problems discovered in early design phases. That such needs have generated interest in automated design of aircraft structures under flutter constraints is evidenced by a growing body of literature, including a comprehensive review by Stroud.[1]

When only a single flutter analysis is performed, it is common practice[2] to use the vibration modes as generalized coordinates for the purpose of reducing the order of the flutter problem. For the design process, the analysis has to be repeated many times; thus there is a tendency to use an invariant set of modes (usually the vibration modes of the initial structure) as generalized coordinates in the flutter calculation, even though the structure is being changed during the process.[3-5] This usage is motivated by the desire to avoid the recomputation of the modes and the aerodynamic forces associated with these modes. The potential disadvantage of this fixed-mode approach is that a larger number of modes may be needed to describe adequately the behavior of a structure which might be changed substantially during the design process than if the modes are continually updated. The use of a large number of modes entails the use of more complicated modes, and this in turn requires more effort (e.g., more collocation or quadrature points) in the computation of the aerodynamic forces.

In comparing the use of fixed modes with the use of updated natural vibration modes in the design process, it is appropriate to distinguish between resizing methods that use derivatives of the flutter parameters (e.g., flutter speed or altitude) with respect to structural parametes, and methods that do not use such derivatives. Many mathematical programing methods of optimization[6,7] require such derivatives whereas optimality criteria methods[8] often do not.

The present work examines the use of periodically updated natural modes (called "changing modes" hereafter) vs fixed modes in the design process. The first part of this work deals with "no-derivatives" methods. In order to make the computations as efficient as possible, it is desirable to minimize the amount of computing that must be repeated each time the structure is changed and maximize the portion of the work that is independent of structural changes; hence done once for all. Therefore, the generalized aerodynamic forces required, as well as the flutter solution process itself, are separated into parts that are mode-dependent and parts that are mode-independent. On the basis of this separation, an attempt is made to identify the parameters that dictate whether or how often the modes should be updated.

The second part of this work deals with derivative methods. In order to establish a basis for the comparison between use of fixed and changing modes with such methods, the convergence of these derivatives with increasing number of modes is examined for both types of modes. Very poor convergence is found in some examples, and this situation frustrates the attempt to compare the use of fixed and changing modes for resizing methods, which employ derivatives. This poor convergence is investigated and shown to be similar to the poor convergence that may occur for derivatives of a Fourier Series.

Flutter Calculations

Flutter Equation

The matrix form of the flutter equation for a finite-element model of a structure is

$$[\bar{K}(I+i\bar{G}) - \omega^2\bar{M}+q\bar{A}]\,w=0 \tag{1}$$

[compare Eq. (35) of Ref. 9] where $\bar{K}$, $\bar{G}$, $\bar{M}$, and $\bar{A}$ are the stiffness, structural damping, mass, and aerodynamic matrices, respectively; and w is a vector of displacements at the nodes of the finite-element model. The order of Eq. (1), which is the number of degrees of freedom of the finite-element model, is usually large and is denoted by m. To reduce the order of the problem, a number of structural modes $z^1, z^2,\ldots z^n$, $(n<<m)$ are used as generalized coordinates, that is

$$w=Zc \tag{2}$$

c being the complex amplitude of the modes. Equation (2) is used to transform Eq. (1) into

$$[K(I+iG)-\omega^2M+qA]\,c=0 \tag{3}$$

where

$$K=Z^T\bar{K}Z \tag{4a}$$

$$G=K^{-1}Z^T\bar{K}\bar{G}Z \tag{4b}$$

$$M=Z^T\bar{M}Z \tag{4c}$$

$$A=Z^T\bar{A}Z \tag{4d}$$

For a given aerodynamic configuration, the aerodynamic influence coefficient (AIC) matrix $\bar{A}$ is a function of the Mach number M, and reduced frequency k only, whereas the generalized-aerodynamic-force (GAF) matrix A is a function of k, M, and the modes z^i.

Calculation of the Generalized Aerodynamic Forces

The matrices K and M may be calculated efficiently from Eqs. (4a) and (4c) because $\bar{K}$ and $\bar{M}$ real, symmetric, and sparse. On the other hand, $\bar{A}$ is complex, asymmetric, and full, and so it is usually costly to calculate and store for problems with many degrees of freedom (d.o.f). Moreover, use of Eq. (4d) usually involves approximations such as lumped forces and discrete deflections. Hence, rather than calculate A directly from Eq. (4d), it is more accurate and more efficient to derive an analogous expression from the integral definition of generalized aerodynamic forces (GAF) in terms of modal displacements and loads. Thus, for a wing-type structure lying near the x,y plane, the GAF are[9]

$$A_{ij}(k,M)=\int_S\!\!\int \frac{\Delta p^j(x,y)}{q}\,h^i(x,y)\;\mathrm{d}x\mathrm{d}y \tag{5}$$

where Δp^j is the pressure difference between the upper and lower surfaces per unit amplitude of the jth modes, and $h^i(x,y)$ are the z-displacements per unit amplitude of the ith mode. The vector z^i is the displacements of the ith mode at the structural grid, and $h^i(x,y)$ is obtained from it by interpolation; that is

$$h^i(x,y)=t^T(x,y)\,z^i \tag{6}$$

where $t(x,y)$ is a suitable interpolation function.

The calculation of the GAF using Eq. (5) is exemplified for kernel function aerodynamics in Appendix A. It is shown that the GAF matrix A can be written as

$$A=H^TUB^{-1}W \tag{7}$$

The parameters in the GAF matrix which will vary during the design process are reduced frequency k and structural modes Z^i. The functional dependence of the component matrices in Eq. (7) on these parameters is indicated in Table 1.

Note that the dependence of W on reduced frequency is trivial since k appears only as a multiplying factor in the imaginary part of W. The only other frequency-dependent component is B^{-1}, which is calculated once for all. The matrices U and B^{-1} are independent of the modes, whereas H^T and W are matrices that are derived from the values of the modes at collocation or quadrature points. Another form of separation of the GAF into mode-dependent and mode-independent parts may be found in Ref. 10.

Equation (7), which was used for the computations shown herein, is thus of form smiliar to that of Eq. (4d). That is, U B^{-1} appears as a mode-independent pseudo-AIC matrix, which is pre- and post-multiplied by matrices defined by

Table 1 Dependence of GAF component matrices on k and z^i

Matrix	H^T	U	B^{-1}	W
Function of k	no	no	yes	trivial
Function of z^i	yes	no	no	yes

modal deflections.‡ Equation (7), however, was obtained from global considerations; i.e., quadrature implementation of the integral over the wing surface [Eq. (5)], in which both Δp^j and h^i are treated as continuous functions of x and y. Equation (4d), on the other hand, reflects local relations; i.e., the elements of $\bar{A}$ relate deflection at one location on the wing to lumped load at another location. Thus, as usually used, Eq. (4d) is equivalent to evaluating the integral in Eq. (5) by treating the deflection h^i (and sometimes even Δp^j) as constant over the area element associated with the individual term in Eq. (4d) and is, therefore, analogous to rectangular integration. Consequently, for matrices of comparable size. Eq. (7) should be the more accurate formulation; conversely, for a given level of accuracy, the matrices required in Eq. (7) could be considerably smaller than those in Eq. (4d).

Flutter Solution

Equation (3) represents a complex, nonlinear eigenvalue problem in the three real parameters ρ, M, and k. Usually, either the Mach number or the density is fixed, and the eigenproblem is solved for the other two parameters. One of the popular processes for solving the nonlinear eigenproblem [Eq. (3)] is known as the *V-g* method (see Ref. 9 for details). This method requires the solution of a series of linear eigenproblems for assumed values of k in order to determine the flutter condition where the frequency ω is real. In this method, the GAF are required for a large number of k values. It is common[2] to calculate the GAF at a relatively small number of k values and then interpolate them for the additional points. When a close approximation for the flutter speed and frequency is available, methods that are faster than the V-g method are available.[6] However, such methods are suitable only when design changes between iterations are small, and even then, there is a danger of discontinuity in the flutter parameters with respect to changes in design variables.[11] For no-derivative methods, for example, it is assumed herein that iterative steps in the design process are large, and the V-g method must be used for each iteration. The four main steps in the flutter solution process are, therefore, as follows: 1) calculation of vibration modes; 2) calculation of mode-independent aerodynamic matrices [$\boldsymbol{UB}^{-1}$ in Eq. (7)] for a aeries of reduced frequencies, k; 3) calculation of the GAF for the same reduced frequencies and interpolation at additional points; and 4) V-g solution.

No-Derivative Methods

Breakdown of Computer Time

For resizing methods that do not require derivatives of the flutter parameters, a major part of the computational effort is the repeated flutter analysis. In the following, computer time breakdown for the four steps in the flutter calculation is given for one analysis. This time breakdown is used to assess the relative efficiency of using fixed vs changing modes by noting which parts of the analysis must be repeated in each case.

Two examples of actual computer time breakdown for flutter analysis are given for the wings shown in Fig. 1. The results were obtained by use of the computer code WIDOWAC (WIng Design Optimization With Aeroelastic Constraints),[12] which uses a kernel-function method for subsonic aerodynamics and piston theory for supersonic aerodynamics. Both wings are modeled with membrane-cover panels and shear-web finite elements. The wing structure is assumed to have no camber or twist, and only the upper half of the wing is analyzed; the wings are assumed to be clamped at the root. The wing in Fig. 1a is a full-depth sandwich structure, where the sandwich core is modeled by very stiff shear webs. The structural grid consists of 38 nodes (93 unconstrained d.o.f.); and 16 collocation points, along with 16 pressure functions, are used in the aerodynamic calculations. The wing in Fig. 1b is a built-up structure modeled with 70 structural nodes (207 unconstrained d.o.f.) and 36 collocation points for the aerodynamic calculations. For both wings, the GAF are calculated for $M=0.6$ at 10 k values and interpolated for an additional 81 values, so that 91 linear eigenproblems are solved. The complexity of these structural and aerodynamic models is considered to be typical of models used for flutter analysis in preliminary design.

The time breakdown for four, six, and eight modes is shown in Fig. 2 for the full-depth sandwich wing. Of the four operations shown in the figure, the calculation of the mode-independent aerodynamic matrices is most time-consuming, but need not be repeated. The calculations of the modes and GAF are done only once for fixed modes, but have to be repeated for changing modes. The V-g solution has to be repeated for both fixed and changing modes.

The computer time breakdown for the built-up wing is given in Fig. 3. The larger number of collocation points used for this wing (compared to the sandwich wing) results in a larger computation time for the aerodynamic matrices, and the larger number of d.o.f. results in a larger computation time for the modes. The computation time for the GAF is affected by both parameters, whereas the V-g solution is affected by neither, and therefore is the same in Figs. 2 and 3.

Several observations may be made from the data presented in Figs. 2 and 3. First, the recalculation of the GAF for a new set of modes takes a small part of the total flutter solution time. Second, a small change in the number of modes may be very important in terms of overall computation time for a reanalysis. Third, the computation time required for calculation of the modes and GAF is sensitive to the number of d.o.f., and these calculations are done once for fixed modes but repeated for changing modes. These observations are based on the results of a particular computer program, but they should be typical of other design-oriented programs. When computations are made with a general-purpose analysis program that is not design oriented, these trends might be different.

Choice of Modes

The choice of fixed vs changing modes hinges on the convergence properties of the flutter solution when fixed modes are used for a structure being continually changed during the iterative design process. If convergence is comparable for the two types of modes, there appears to be no reason to use changing modes. However, if the flutter solution converges with fewer natural modes, the choice will depend on the tradeoff between the cost of updating the modes and the cost of using more fixed modes (see Figs. 2 and 3). In the case of changing modes, the major question is how often to update the modes. On the basis of the data in Fig. 2 and 3, the following observations are offered. 1) *Number of d.o.f. of the dynamic model:* If the number of d.o.f. is very large, modes should be updated only infrequently, because the natural modes would be expensive to calculate; 2) *Phase of design:* If small changes in the structure are anticipated, fixed modes should be better; if large changes are anticipated, frequently updated modes should be better because a large number of fixed modes would be needed for good representation of all of the structures in the design process.

The crucial factor determining the frequency of updating the modes is the change of the flutter parameters when the

‡If the downwash matrix $\boldsymbol{W}$ is expressed in terms of a differentiating matrix $\boldsymbol{D}$ operating on the matrix of modal displacements $\boldsymbol{H}$(i.e., $\boldsymbol{W}=\boldsymbol{DH}$) than Eq. (7) can be written as $\bar{A}=\boldsymbol{H}^T\boldsymbol{UB}^{-1}\boldsymbol{DH}$ which is in exactly the same form as Eq. (4d) with the product $\boldsymbol{UB}^{-1}$ $\boldsymbol{D}$ playing the role of the AIC matrix $\bar{A}$.

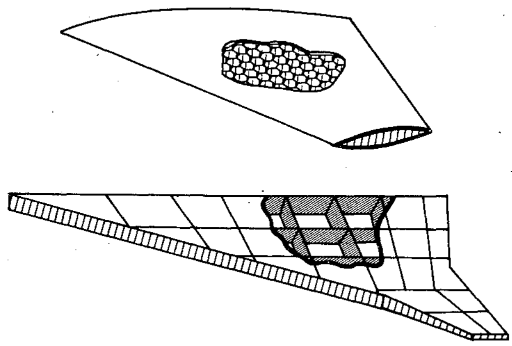

Fig. 1 Description of wings. a) full depth sandwich wing, 93 d.o.f., 16 collocation points; b) built-up arrow wing, 207 d.o.f., 36 collocation points.

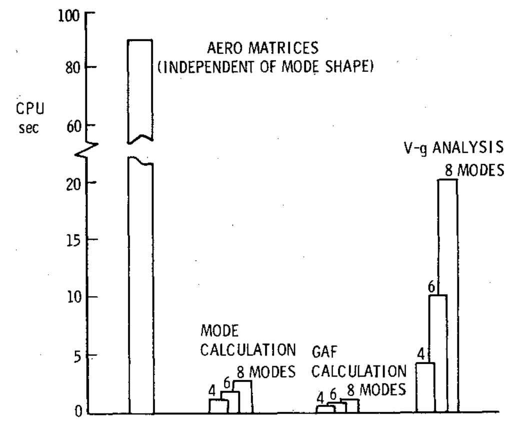

Fig. 2 Computation time breakdown for flutter analysis for full-depth sandwich wing. GAF calculated at 10 reduced frequencies, interpolated at 81 reduced frequencies.

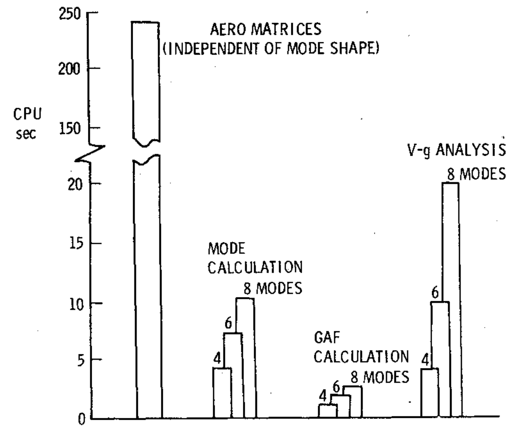

Fig. 3 Computation time breakdown for flutter analysis for built-up arrow wing. GAF calculated at 10 reduced frequencies, interpolated at 81 reduced frequencies.

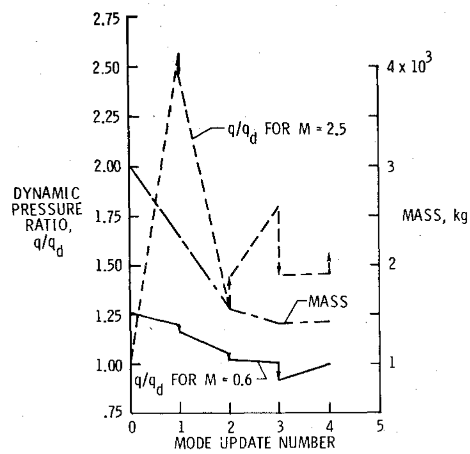

Fig. 4 History of mass and flutter dynamic pressure.

modes are updated. If the changes are too large, the optimization process may be disrupted. If modes are updated only a few times, a larger number of modes is required, as shown by the following example.

The full depth sandwich wing (Fig. 1) is optimized in Ref. 12 (sample problem 2) under two flutter constraints: $M=0.6$ at an altitude of 1524 m (5000 ft), and $M=2.5$ at an altitude of 7620 m (25,000 ft). Five continually updated modes were used. The same wing subsequently has been optimized by use of modes updated four times during the optimization process (at the end of each unconstrained optimization), using 5, 8, 10, 15, and 20 modes. The five-mode optimization broke down because of the large changes in flutter speed when the modes were updated. The history of the normalized flutter dynamic pressures and wing structural mass is shown in Fig. 4 for the eight-mode case. The vertical arrows indicate the magnitude of the changes when the modes were updated. Figure 5 shows the maximum change in dynamic pressure due to mode updating as a function of the number of modes. Note that, for eight modes, the maximum change in the more critical subsonic dynamic pressure is less than 10%. For the supersonic flutter condition, 10 or 12 modes would be required for comparably small change. Although this example cannot be used to obtain general conclusions, it demonstrates the tradeoff between number of modes and number of updatings.

Derivative Methods

Derivative methods, such as most mathematical programing techniques, require the derivatives of the flutter constraints with respect to the design variables. These derivatives may be used for determining how to resize the structure. Moreover, the derivatives offer a means for approximating flutter parameters without a reanalysis when a structure is resized. Usually, flutter constraints are expressed as a restriction on the flutter speed or the flutter altitude; accordingly, the derivatives of these parameters usually are needed for derivative methods. It is known that the flutter speed or altitude may vary discontinuously with design parameters, and alternative flutter constraints that are based on continuous parameters of the flutter phenonemon have been proposed.[7] Presently, however, almost all derivative methods use the derivatives of the flutter speed or altitude, and the present study is limited to the consideration of such methods.

Expressions for the derivatives of the flutter speed and frequency are given in the literature.[3,5,13,14] These expressions are based on the implicit assumption that the vibration modes are fixed. Expressions for the derivatives of the flutter altitude, reduced frequency, and Mach number are obtained in Ref. 15 for the more general case in which the modes are functions of the design variables.

The computation time required to obtain all of the derivatives may be larger than that required for a flutter analysis, depending on the number of design variables and on the method used for obtaining the derivatives. Therefore, the data presented in Figs. 2 and 3, although applicable also to the flutter analysis for derivative methods, are not sufficient for comparing the merits of fixed and changing modes, since calculation of the derivatives also must be considered.

Convergence of Derivatives

Any comparison between fixed and changing modes for derivative methods hinges on the number of modes which are

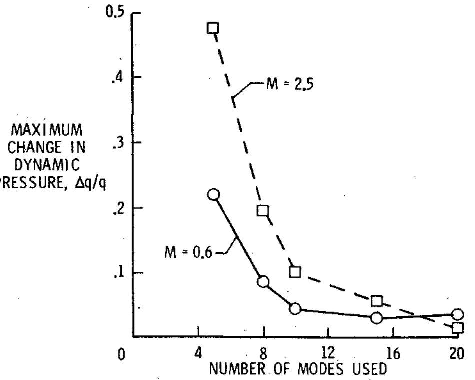

Fig. 5 Dependence of maximum change in dynamic pressure on number of modes.

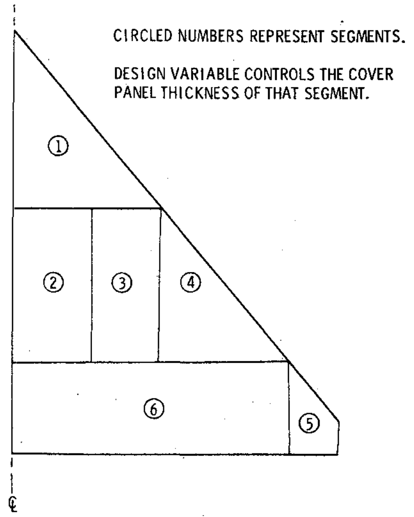

Fig. 6 Definition of design variables for full depth sandwich wing.

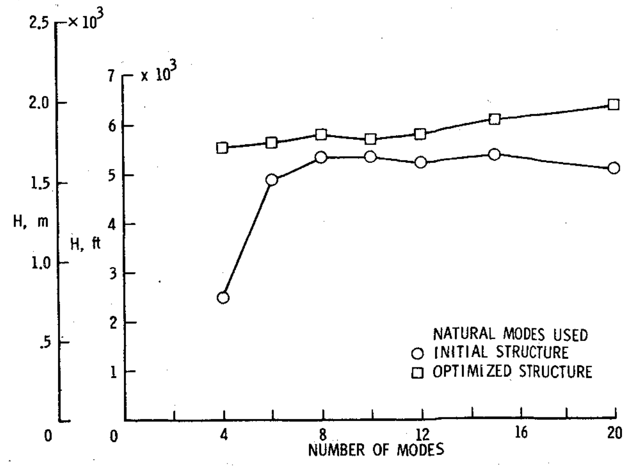

Fig. 7 Flutter altitude for full-depth sandwich wing optimized structure at $M = 0.6$.

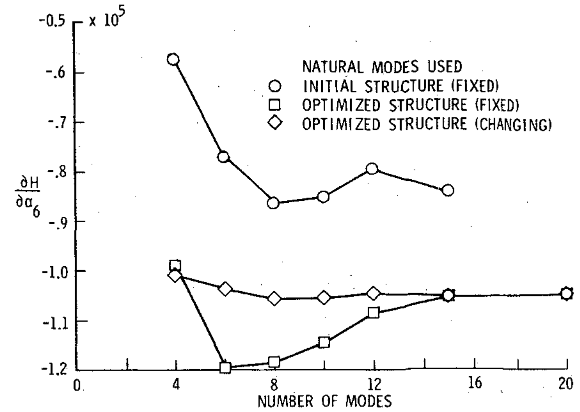

Fig. 8 Derivative of flutter altitude with respect to design variable 6 for full-depth sandwich wing at $M = 0.6$.

needed to calculate the derivatives of the flutter speed or altitude with respect to a design variable. For this reason, the convergence of the derivatives with increasing number of modes is first investigated for both methods. Assume that the fixed modes are the vibration modes of the initial structure. Thus, the flutter speed calculated for the initial structure will be the same for fixed and changing modes, because the same set of generalized coordinates, the vibration modes of the initial structure, are used in the calculations. Of course, for subsequent analyses, different generalized coordinates, the initial vibration modes (fixed modes), and the updated vibration modes (changing modes), would be used in the calculations. In contrast to the flutter speeds for the initial structure, the derivatives for fixed and changing modes differ since the fixed-mode derivatives do not include the effects of the derivatives of the modes with respect to the design variables, whereas for changing modes this effect is included. The number of modes needed for convergence of the flutter speed may be different from the number needed for the convergence of a derivative of the flutter speed with respect to a structural parameter. This situation was illustrated in Ref. 15 by an example of a two-dimensional wing-flap system.

Another example of convergence of the flutter speed and its derivatives is given for the full-depth sandwich wing shown in Fig. 1. The WIDOWAC program[12] was used to calculate the flutter altitude and its derivatives with respect to six design variables that described the thickness of the cover panels (see Fig. 6). Kernel-function aerodynamics was used for the calculations, and the derivatives were calculated by forward-difference approximations. The results are given in Figs. 7 and 8. Again, the natural modes were used both as fixed and changing modes.

Figure 7 shows that as few as four undamped natural modes of the optimized structure are sufficient to yield a reasonable estimate of the flutter altitude for that structure, whereas about twice that number of modes of the initial structure are needed. In connection with Fig. 7, note that altitude is a much more sensitive flutter parameter than speed or density. For example, the difference between the two curves for 20 modes represents a difference of less than 5% in flutter dynamic pressure.

In Fig. 8, one of the larger (and hence one of the more influential) flutter derivatives for the wing of Fig. 6 is shown as a function of the number of modes used. Use of the natural modes of the optimized structure, including rate of change of mode shape with respect to design variable (diamond symbol), produces convergence of the derivative with about eight modes. If the derivative of mode shape is not included, however, (square symbol), the flutter derivative still converges, but at least 15 modes are required. If the modes of the initial structure are used to describe the flutter motion of the optimized wing (circle symbol), the flutter derivative appears to be converging very slowly, if at all. The poor convergence properties of the derivatives for fixed modes frustrates the attempt to compare the efficiency of using fixed vs changing modes for resizing methods that employ derivatives.

Explanation of Poor Convergence

An explanation for the poor convergence of the derivatives with increasing number of vibration modes is furnished by examination of the equations for the derivatives.[15] From Eq.

(A28) of Ref. 15

$$L_k \frac{dk}{d\alpha} + L_M \frac{dM}{d\alpha} + L_\rho \frac{d\rho}{d\alpha} = R \quad (8)$$

where

$$L_k = c^{*T}\left[-\frac{2\omega^2}{k} M + q\frac{\partial A}{\partial k}\right]c \quad (9)$$

$$L_M = c^{*T}\left[-\frac{2\omega^2}{M} M + q\left(\frac{2}{M}A + \frac{\partial A}{\partial M}\right)\right]c \quad (10)$$

$$L_\rho = c^{*T}\left[-\frac{2\omega^2}{a}\frac{da}{d\rho} M + q\left(\frac{1}{\rho} + \frac{2}{a}\frac{da}{d\rho}\right)A\right]c \quad (11)$$

and

$$R = c^{*T}\left[\frac{dK}{d\alpha}(I + iG) - \omega^2\frac{dM}{d\alpha} + q\frac{\partial A}{\partial \alpha}\right]c \quad (12)$$

Equation (8) is a complex equation for the three unknowns $dk/d\alpha$, $dM/d\alpha$, and $d\rho/d\alpha$. Usually, one of the three is taken to be zero (e.g., for flutter at constant Mach number, $dM/d\alpha = 0$), and the other two are found from Eq. (8).

Consider an example in which the design variable is Young's modulus, designated for this structure in the form $Y = Y_0(1+\alpha)$, where subscript zero denotes the initial structure, and α is the design variable. Thus, $dM/d\alpha = 0$, and $K = K_0(1+\alpha)$. The stiffness matrix K_0 is diagonal because the modes are natural vibration modes. Further, if the modes are normalized so that $M = I$, then the diagonal elements of K_0 will be the squares of the modal frequencies. Under these conditions, the natural modes, and hence the GAF, are independent of α. If the structural damping is neglected, then Eq. (12) for R becomes

$$R = c^{*T}\frac{dK}{d\alpha}c = c^{*T}K_0 c = \sum_{i=1}^{n} c_i^* c_i \omega_i^2 \quad (13)$$

where c is the flutter eigenvector, and c^* is the left flutter eigenvector. The quantity R may be expected to converge very slowly with increasing n, because ω_i^2 increases with i. In as much as the derivatives are obtained from Eq (8), slow convergence of R implies slow convergence of the derivatives. The expected slow convergence is demonstrated here only for the special case of derivative with respect to Young's modulus. It may be expected, however, that terms like those appearing in Eq. (13) are present in the equation for R in more general cases. These slowly converging terms depend only on structural terms and, therefore, the problem may be expected to exist for any type of aerodynamics.

Recognizing that the use of vibration modes as generalized coordinates is a generalization of the use of trigonometric functions for the same purpose also helps in understanding the convergence problem. Thus, if a function $f(x)$ is expanded in a sine series

$$f(x) = \sum_l a_l \sin lx \quad (14)$$

then

$$f'(x) = \sum_l l a_l \cos lx \quad (15)$$

and the derivative converges (if at all) more slowly than the function itself.

Implications of Poor Convergence

The implication of the poor convergence of the derivatives of flutter parameters is that such derivatives should be used with caution unless the flutter solution converges rapidly with increasing number of modes. Some quantitative measure of the desired convergence properties is afforded by Eq. (13); from this equation it appears that a criterion for convergence of the derivatives may be

$$\lim_{i\to\infty} |ic_i c_i^* \omega_i^2| = 0 \quad (16)$$

which is a rather severe convergence condition. The criterion expressed by Eq. (16) has been obtained for a particular type of stiffness variation, and is therefore neither a sufficient nor a necessary condition for convergence in general. Moreover, it is possible that the first few terms of a series may furnish a useful approximation to a function expressed by the series, even if the series does not converge.

It is certain, however, that if many modes are needed for convergence of the flutter solution, then the c_i's in Eq. (13) converge slowly, and the derivatives converge even more slowly if at all. It is thus concluded that, when derivative methods are used, it is desirable to use a set of vibration modes which causes the flutter solution to converge rapidly. It is possible that the need for rapid convergence means a need for frequently updated natural modes.

The need to use frequently updated modes does not necessarily imply a need to calculate derivatives based on changing modes. Although the example in Fig. 8 shows that derivatives that take into account the change in the modes may converge faster, it is not clear at this point whether or not the faster convergence is counterbalanced by the extra computation needed to obtain the derivatives of the modes.

Conclusions

Some aspects of efficient modal flutter analysis for use in aircraft structural design that may involve many iterations were investigated. Expressions for the generalized aerodynamic forces were derived that are separated into mode-dependent and mode-independent parts, in order to permit rapid recalculations of the forces when the modes are changed. The computer time breakdown for the various parts of the flutter analysis was obtained for some example problems. These examples were used to compare the efficiency of using continually updated natural modes vs fixed modes in the design process. From the examples, it appears that the main computational penalty in using updated natural vibration modes in the design process lies in the recalculation of the modes, rather than in the recalculation of the generalized aerodynamic forces.

The choice of fixed vs changing modes hinges on the convergence properties of the flutter solution when fixed modes are used for a structure being continually changed during the iterative design process. If convergence is comparable for the two types of modes, there appears to be no reason to use changing modes. However, if the flutter solution converges faster using natural modes, the choice will depend on the tradeoff between the cost of updating the modes and the cost of using more fixed modes. The use of periodically updated modes probably would be more efficient than the use of fixed modes in the preliminary design stage (small number of degrees of freedom and possibly large structural changes). In more advanced design stages (large number of degrees of freedom and possibly small structural changes), fixed modes may be satisfactory when used with no-derivative resizing methods.

The preceding comparisons are based on time required for analysis and are applicable to resizing methods that do not require the derivatives of any of the flutter parameters with respect to structural variables. Many resizing methods, however, do require the derivatives of the flutter frequency, flutter speed, or flutter altitude with respect to design variables. The convergence of such derivatives with increasing number of modes was investigated. It was found that the derivatives converge more slowly than the flutter solution and may exhibit very poor convergence characteristics. Therefore, when derivative methods are used for resizing the structure, it is desirable to use a set of modes which causes the flutter solution to converge rapidly, since a slowly converging flutter solution probably will lead to unacceptable convergence characteristics for the derivatives.

Appendix A: Derivation of Equations

Calculation of the Generalized Aerodynamic Forces for Analysis

The generalized aerodynamic forces are given in Eq. (5) of the text as

$$A_{ij}(k,M) = \int_S\int \frac{\Delta p^j(x,y)}{q} h^i(x,y)\,\mathrm{d}x\mathrm{d}y \qquad \text{(A1)}$$

The lifting pressure Δp^j is assumed in the following to be obtained from kernel function aerodynamics; that is, from a downwash-pressure integral equation[16]

$$\frac{\bar{w}^j(x,y)}{V} = \int_S\int \frac{\Delta p^j(\xi,\eta)}{q} K[(x-\xi),\ (y-\eta),M,k]\,\mathrm{d}\xi\mathrm{d}\eta \quad \text{(A2)}$$

where $\bar{w}^j(x,y)$, the downwash at (x,y), due to motion in the jth mode, is related to the displacement $h^j(x,y)$ by

$$\frac{\bar{w}^j(x,y)}{V} = \left(\frac{\partial}{\partial x} + ik\right) h^j(x,y) \qquad \text{(A3)}$$

and K is the kernel function, which is known.[17,18]

In order to solve Eq. (A2) for the lifting pressure, it is common to approximate it as a linear combination of pressure functions

$$\frac{\Delta p^j(x,y)}{q} = \sum_{l=1}^{n_p} e_l^j u_l(x,y) \qquad \text{(A4)}$$

and then impose Eq. (A2) at a number of downwash control points (x_t,y_t) which is equal to or larger than the number of pressure functions. Substituting the pressure from Eq. (A4) into Eq. (A2) yields

$$\frac{\bar{w}^j(x,y)}{V} = \sum_{l=1}^{n_p} e_l^j B_l(x,y,M,k) \qquad \text{(A5)}$$

where the pressure-kernel integrals are

$$B_l = \int_S\int u_l(\xi,\eta) K[(x-\xi),(y-\eta),k,M]\,\mathrm{d}\xi\mathrm{d}\eta \qquad \text{(A6)}$$

The coefficients e_l^j are found by enforcing Eq. (A5) at n_c collocation points (x_t,y_t), $t=1(1)n_c$. For the jth mode,

$$\left(\frac{\partial h^j}{\partial x} + ikh^j\right)\Big|_{(x_t,y_t)} = \sum_{l=1}^{n_p} e_l^j B_l(x_t,y_t) \qquad \text{(A7)}$$

which is a system of n_c linear algebraic equations for the n_p pressure coefficients. If n_c is larger than n_p, then the system may be solved by a least-squares method.

In terms of the pressure coefficients and the pressure functions [Eq. (A4)], the generalized aerodynamic forces are

$$A_{ij} = \sum_{l=1}^{n_p} P_l^i e_l^j \qquad \text{(A8)}$$

where the force quadrature coefficients P_l^i are

$$P_l^i = \int_S\int u_l(x,y) h^i(x,y)\,\mathrm{d}x\mathrm{d}y \qquad \text{(A9)}$$

The evaluation of the generalized aerodynamic forces thus proceeds in the following steps:

1. Choose pressure functions [Eq. (A4)] and structural modes.
2. Calculate the pressure-kernel integrals [Eq. (A6)].
3. Solve Eq. (A7) for the pressure-function coefficients for each mode.
4. Calculate the force quadrature coefficients [Eq. (A9)] for each mode.
5. Calculate the generalized aerodynamic Forces from Eq. (A8).

Calculation of the Generalized Aerodynamic Forces for Iterative Design

In order to separate the mode-dependent and mode-independent parts of the calculation, it is convenient to case the previous equations in matrix form. Define

$$W_{tj} = (\partial h^j/\partial x + ikh^j)\,|_{(x_t,y_t)} \qquad \text{(A10)}$$

$$E_{lj} = e_l^j \qquad \text{(A11)}$$

and

$$B_{tl} = B_l(x_t,y_t) \qquad \text{(A12)}$$

Then Eq. (A5) becomes

$$\boldsymbol{W} = \boldsymbol{BE} \qquad \text{(A13)}$$

or

$$\boldsymbol{E} = \boldsymbol{B}^{-1}\boldsymbol{W} \qquad \text{(A14)}$$

When $\boldsymbol{B}$ is not square and Eq. (A13) is solved by a least-squares method, Eq. (A14) is only a symbolic form.

The integration of Eq. (A9) is usually numerical, that is

$$P_l^i = \sum_{r=1}^{n_q} Q_r U_l(x_r,y_r) h^i(x_r,y_r) \qquad \text{(A15)}$$

where (x_r,y_r) is a set of n_q quadrature points, and Q_r are the associated weighting factors. Define

$$U_{rl} = Q_r U_l(x_r,y_r) \qquad \text{(A16)}$$

and

$$H_{ri} = h^i(x_r,y_r) \qquad \text{(A17)}$$

then

$$\boldsymbol{P} = \boldsymbol{H}^T\boldsymbol{U} \qquad \text{(A18)}$$

Equation (A8) in matrix form is

$$\boldsymbol{A} = \boldsymbol{PE} \qquad \text{(A19)}$$

and so, from Eqs. (A18) and (A14),

$$\boldsymbol{A} = \boldsymbol{H}^T\boldsymbol{U}\boldsymbol{B}^{-1}\boldsymbol{W} \qquad \text{(A20)}$$

Equation (A20) is properly separated into mode-dependent and mode-independent parts. The matrix product $\boldsymbol{UB}^{-1}$ is independent of the modes, and hence of the structural parameters. The matrices $\boldsymbol{H}$ and $\boldsymbol{W}$ involve only interpolation of the modes and their slopes at sets of collocation or quadrature points.

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